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Asymptotic Γ-distribution for stochastic difference equations

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We study asymptotic properties of non-negative random variables X_n, $n \geqslant 0$, satisfying the recursion $X_{n+1} = X_n + g(X_n) + \xi_{n+1}$ with $\mathbb{E}(\xi_{n+1} | X_0, \ldots, X_n) = 0$, $\mathbb{E}(\xi_{n+1}^2 | X_0, \ldots, X_n) = \sigma^2(X_n)$. If the functions $g(x)$ and $\sigma^2(x)$ are properly balanced at infinity, X_n is asymptotically Γ-distributed in a suitable scale. This result contains several known theorems.

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asymptotic Γ-distribution * branching processes

1. Introduction and main result

There are several results in probability theory asserting that some sequence of positive random variables is asymptotically Γ-distributed. We present a theorem which contains a few of these results, and give several applications. This is our model: Let $F_0 \subset F_1 \subset \cdots$ be an increasing sequence of σ-fields on some probability space and let $X_0, X_1, \ldots$ be real-valued random variables which satisfy the following assumptions:

(A1) $X_n \geqslant 0$, X_n is F_n-measurable.

(A2) There are functions $g(x)$ and $\sigma^2(x)$, $x \geqslant 0$, and random variables $\xi_1, \xi_2, \ldots$ such that a.s.

$$X_{n+1} = X_n + g(X_n) + \xi_{n+1},$$

$$\mathbb{E}(\xi_{n+1} | F_n) = 0,$$

$$\mathbb{E}(\xi_{n+1}^2 | F_n) = \sigma^2(X_n).$$

(A3) There is a $C > 0$ and a $\delta > 2$ such that for all $n \geqslant 0$ a.s.

$$\mathbb{E}(|\xi_{n+1}|^\delta | F_n) \leqslant C\sigma(X_n)^\delta.$$

Several paper deal with such models (compare [6], [7], [9] and [10]). Some of them are motivated by more specific stochastic processes such as population-dependent branching processes. We shall discuss examples in the next section. We are interested in the behaviour of X_n on the event

$$\mathscr{E}_\infty = \{X_n \to \infty, \text{ as } n \to \infty\}.$$

More specifically we want to investigate the conditional distribution of X_n, given $\mathscr{E}_\infty$. In [6] it was shown that under broad assumptions X_n is asymptotically normal. Here we show that in a more specific situation (the magnitudes of $g(x)$ and $\sigma^2(x)$ have to be properly balanced at infinity), there is also the possibility of a gamma limit. Let $g(x) > 0$ for $x > 0$ and define

$$G(x) = \int_1^x \frac{\mathrm{d}y}{g(y)}.$$

Our result requires a certain amount of regularity:

(A4) $g(x)$ is strictly positive and differentiable for $x > 0$. Furthermore there is a $\lambda \geq 0$ such that $G(x)^\mu$ is ultimately convex for $\mu \notin [0, \lambda]$ and ultimately concave for $\mu \in (0, \lambda)$.

Examples are provided by $g(x) = x^\alpha$ with $\alpha < 1$ and $g(x) = \mathrm{e}^{-x}$. In the first case $\lambda = (1-\alpha)^{-1}$, in the second case $\lambda = 0$. (A4) includes two different classes of functions. In the case $\lambda = 0$, (A4) requires the ultimate convexity on $(Gx)^\mu$ for all $\mu \neq 0$, which roughly means that $G(x)$ is increasing more rapidly than any power of x. In fact then $g(x) = \mathrm{O}(x^{-\alpha})$ for any $\alpha > 0$, as follows from Lemma 2 below. In the case $\lambda > 0$ (A4) is equivalent to $g'(x)x/g(x) \to 1 - \lambda^{-1}$ (compare again Lemma 2). A well-known theorem of Karamata (see [1, p. 273]) states that then $g(x)$ is of the form $x^\alpha L(x)$, where $\alpha = 1 - \lambda^{-1}$ and $L(x)$ is slowly varying.

Our main result deals with the random variables

$$Y_n = G(X_n).$$

Theorem 1. *Let* (A1)-(A4) *be satisfied and let* $\Pr(\mathscr{E}_\infty) > 0$. *If for some* $\beta > 0$ *with* $\beta\lambda < 2$,

$$\frac{\sigma^2(x)}{g^2(x)G(x)} \to \beta, \tag{1.1}$$

as $x \to \infty$, *then for any* $t \geq 0$,

$$\Pr(Y_n \leq tn \mid \mathscr{E}_\infty) \to c \int_0^t x^{2/\beta - \lambda} \operatorname{ex}\left(-\frac{2x}{\beta}\right) \mathrm{d}x,$$

as $n \to \infty$, *where* $c^{-1} = (\frac{1}{2}\beta)^{2/\beta - \lambda + 1}\Gamma(2/\beta - \lambda + 1)$.

This theorem was conjectured in [6]. Klebaner [9] treated the case $g(x) \sim \gamma x^\alpha$, $\alpha < 1$, under stronger moment assumptions. The proof of our theorem is based on the method of moments. The main technical problem is that Y_n may have no moments at all. We shall get around this difficulty by introducing an appropriate truncation procedure. The proof of Theorem 1 is given in Section 4, Section 3 contains several auxiliary results. Examples are given in Section 2.

Comments. (1) Note that the number λ appears in the assertion of our theorem, thus assumption (A4) is not of a purely technical character. Also (A3) cannot be removed without any compensation. This follows from the last example in the next section.

(2) The condition $\beta\lambda < 2$ in Theorem 1 is closely connected to the property $\Pr(\mathscr{E}_\infty) > 0$. In fact we may state the following:

Claim. Let (A1)-(A4) be satisfied and assume (1.1) for some $\beta \geq 0$:

(i) If $\lambda\beta > 2$, then $\Pr(\mathscr{E}_\infty) = 0$.

(ii) If $\lambda\beta < 2$, then either $\Pr(\mathscr{E}_\infty) > 0$ or there is a $C > 0$ such that a.s. $\sup_n X_n \leq C$.

For the proof note that in view of Lemma 2 below

$$\frac{\sigma^2(x)}{xg(x)} \to \beta\lambda \quad \text{and} \quad g(x) = O(x^\alpha)$$

for some $\alpha < 1$. Now our claim follows from Theorem 1(ii) and Theorem 2(ii) in Kersting [7].

(3) A functional version of our weak convergence theorem is valid, too. Define stochastic processes $Y_n(t)$, $t \geq 0$, by

$$Y_n(t) = \frac{1}{n} Y_k, \quad \text{if } \frac{k}{n} \leq t < \frac{k+1}{n}.$$

Then the distribution of $Y_n(\cdot)$, conditioned on $\mathscr{E}_\infty$, converges in the Skorohod topology to the distribution of the process Z_t, $t \geq 0$, satisfying the Ito equation

$$dZ_t = (1 + \tfrac{1}{2}(1-\lambda)\beta)\, dt + (\beta Z_t)^{1/2}\, dW_t, \quad Z_0 \equiv 0.$$

Here, W_t, $t \geq 0$, denotes a standard Brownian motion. Since the proof requires additional efforts, we shall not go into it.

2. Examples

2.1. Symmetric random walk

In this example we encounter a connection to the central limit theorem. Let S_n, $n \geq 0$, be a simple random walk on the lattice $\mathbb{Z}^d$, i.e. the transition probabilities are $P_{xy} = (2d)^{-1}$, if $|x-y| = 1$, $P_{xy} = 0$ otherwise. Define $X_n = |S_n|^2$. Then

$$X_{n+1} = X_n + 1 + \xi_{n+1},$$

where

$$\xi_{n+1} = 2\langle S_n, S_{n+1} - S_n \rangle.$$

$\langle\ ,\ \rangle$ denotes the ordinary scalar product. It turns out that

$$\mathbb{E}(\xi_{n+1}\,|\,S_0,\dots,S_n)=0,$$

$$\mathbb{E}(\xi_{n+1}^2\,|\,S_0,\dots,S_n)=\frac{4}{d}X_n,$$

therefore (A2) is satisfied with $g(x)=1$, $\sigma^2(x)=4x/d$. (A4) holds with $\lambda=1$. Also (1.1) is valid with $\beta=4/d$. The condition $\lambda\beta<2$ is equivalent to $d\geq 3$, which is well-known to imply transience of the random walk. Then $\Pr(\mathscr{E}_\infty)=1$, and in view of Theorem 1 X_n/n has an asymptotic Γ-distribution. This follows also from the central limit theorem: $\sqrt{d/n}\,S_n$ is asymptotically standard normal, thus dX_n/n is asymptotically χ^2-distributed with d degrees of freedom.

For an example of a random walk, to which our theorem applies with $\lambda\neq 1$, compare Guivarc'h et al. [2], in particular Chapter VI, Theorem 42.

2.2. Branching process with immigration

This model has attained some attention. Let η_{kn} and θ_n, $n\geq 0$, $k\geq 1$, be independent copies of $\mathbb{N}_0$-valued random variables η respectively θ. Define

$$X_{n+1}=\sum_{k=1}^{X_n}\eta_{kn}+\theta_n,\qquad X_0\equiv 1.$$

Usually X_n is regarded as representing the magnitude of some random population at generation n. In this interpretation η_{kn} is the number of offsprings of the kth member in the nth generation, and θ_n is the amount of immigration into the nth generation. It is not difficult to see that (A2) is fulfilled with

$$g(x)=(\mu-1)x+\nu,\qquad \sigma^2(x)=\rho^2x+\tau^2,$$

where

$$\mu=\mathbb{E}\eta,\quad \nu=\mathbb{E}\theta,\quad \rho^2=\operatorname{Var}\eta,\quad \tau^2=\operatorname{Var}\theta.$$

Note that in the critical case $\mu=1$, (A4) is valid with $\lambda=1$, also (1.1) holds with $\beta=\rho^2/\nu$. In view of Theorem 1, X_n/n has an asymptotic Γ-distribution, if $\rho^2<2\nu$. This is in accordance with known results (compare [5]).

2.3. Population-dependent branching models

Now let $\eta_{kn}(x)$, $n\geq 0$, $k, x\geq 1$, be independent copies of $\mathbb{N}_0$-valued random variables $\eta(x)$. Define

$$X_{n+1}=\sum_{k=1}^{X_n}\eta_{kn}(X_n),\qquad X_0\equiv 1.$$

The interpretation is as in the last example, however, now the number of offsprings of some individuum has a distribution, which may depend on the magnitude of the population. Again (A2) is satisfied with

$$g(x)=x(\mathbb{E}\eta(x)-1),\qquad \sigma^2(x)=x\operatorname{Var}\eta(x),$$

and, if $\mathbb{E}\eta(x)$ and Var $\eta(x)$ are properly adjusted, X_n may again have an asymptotic Γ-distribution. In particular results of Höpfner [4] and Klebaner [8] are contained in Theorem 1.

2.4. *A counterexample*

Finally we give an example, which shows that (A3) cannot be removed from the assumptions of Theorem 1. Let $\eta_0, \eta_1, \ldots$ be independent copies of some $\mathbb{R}^+$-valued random variable η with distribution function $H(x)$ and finite second moment. Define

$$X_n = \max(\eta_0, \ldots, \eta_n).$$

Then we have

$$X_{n+1} = X_n + g(X_n) + \xi_{n+1},$$

where

$$\xi_{n+1} = \max(0, \eta_{n+1} - X_n) - g(X_n)$$

and

$$g(x) = \mathbb{E}\max(0, \eta - x) = \int_x^\infty (1 - H(y))\,\mathrm{d}y.$$

If η has unbounded support, $g(x) > 0$ for all $x > 0$ and $\Pr(\mathscr{E}_\infty) = 1$. Further (A2) is satisfied with $F_n = \sigma(\eta_0, \ldots, \eta_n)$ and

$$\begin{aligned}\sigma^2(x) &= \mathbb{E}\max(0, \eta - x)^2 - g(x)^2\\ &= 2\int_x^\infty g(y)\,\mathrm{d}y - g(x)^2 = 2\int_x^\infty g(y)H(y)\,\mathrm{d}y.\end{aligned}$$

Now let us assume that (A4) is satisfied for some $\lambda \geq 0$ (f.e. let $H(x) = 1 - x^{-1/\lambda}$ for $x \geq 1$). It follows from Lemma 2 below that

$$g'(x)G(x) \to \lambda - 1.$$

Using l'Hospital's rule we obtain

$$\frac{\sigma^2(x)}{g^2(x)G(x)} \sim \frac{-2g(x)H(x)}{g(x) + 2g(x)g'(x)G(x)} \sim \frac{2}{1 - 2\lambda},$$

thus (1.1) is satisfied with $\beta = 2(1 - 2\lambda)^{-1}$. Note that necessarily $\lambda < \frac{1}{2}$. The requirement $\beta\lambda < 2$ coincides with $\lambda < \frac{1}{3}$. However, the conclusion of Theorem 1 is not valid in this example. First note that

$$-g'(X_n) = \min(1 - H(\eta_0), \ldots, 1 - H(\eta_n)).$$

Now $1 - H(\eta)$ is uniformly distributed on $[0, 1]$, thus for any $s \geq 0$,

$$\Pr\left(-g'(X_n) \geq \frac{s}{n}\right) = \left(1 - \frac{s}{n}\right)^n \to \mathrm{e}^{-s}.$$

Since $-g'(x)G(x)=1-\lambda+o(1)$, this entails

$$\Pr(Y_n \leqslant nt) \to \exp\left(-\frac{1-\lambda}{t}\right),$$

such that Y_n/n is not asymptotically Γ-distributed. The reason is that (A3) is not fulfilled in this example.

3. Auxiliary results

In this section we provide a few technical results. Let us begin with some analytical statements.

Lemma 2. *Let $g(x)$ be strictly positive and differentiable for $x>0$.*

(i) (A4) *is equivalent to*

$$g'(x)G(x) \to \lambda-1,$$

as $x\to\infty$. Furthermore (A4) *implies*

$$\frac{g(x)G(x)}{x} \to \lambda$$

and

$$g(x)=O(x^{1-1/\mu}) \quad \textit{for any } \mu>\lambda.$$

(ii) *If $\lambda>0$,* (A4) *is equivalent to*

$$\frac{g'(x)x}{g(x)} \to 1-\lambda^{-1}.$$

(iii) *If additionally to* (A4), (1.1) *holds, then*

$$\frac{\sigma^2(x)}{xg(x)} \to \lambda\beta.$$

Proof. (i) (A4) states that the second derivative of $G(x)^\mu$, namely

$$\mu g(x)^{-2}G(x)^{\mu-2}(\mu-1-g'(x)G(x)),$$

has to be ultimately positive for $\mu>\lambda$ and $\mu<0$ and ultimately negative for $0<\mu<\lambda$. Obviously this is equivalent to $g'(x)G(x)=\lambda-1+o(1)$. The second claim follows from

$$g(x)G(x)=\int_1^x (g'(x)G(x)+1)\,\mathrm{d}y.$$

Furthermore note that for any $\mu>\lambda$ in view of (A4), $G(x)^\mu \geqslant x$ for large x. Since $g(x)=O(x/G(x))$, the third claim follows.

(ii) Let $\lambda > 0$. If (A4) holds, then in view of (i),

$$\frac{g'(x)x}{g(x)} = g'(x)G(x)\frac{x}{g(x)G(x)} \to \frac{\lambda-1}{\lambda}.$$

Conversely, if $xg'(x)/g(x) \to 1-\lambda^{-1}$, by means of l'Hospital's rule

$$\frac{x/g(x)}{G(x)} \sim \frac{1/g(x) - xg'(x)/g^2(x)}{1/g(x)} \to \lambda^{-1},$$

thus, using $g'(x)x/g(x) \to 1-\lambda^{-1}$ again

$$g'(x)G(x) \sim (1-\lambda^{-1})\frac{g(x)G(x)}{x} \to \lambda - 1,$$

and (A4) follows in view of (i).

(iii) This is a direct consequence of (i). □

Lemma 3. *Let* (A4) *be satisfied. Then there is a* $C > 0$ *such that for* $\varepsilon > 0$ *small enough,* $x \geq 2$ *and* $|a| \leq \varepsilon g(x)G(x)$,

$$\left|\frac{g(x+a)}{g(x)}\right| + \left|\frac{G(x+a)}{G(x)}\right| \leq \frac{C|a|}{g(x)G(x)} \leq C\varepsilon.$$

Proof. In view of Lemma 2, $x - \varepsilon g(x)G(x) \to \infty$, as $x \to \infty$, if only $\varepsilon > 0$ is small enough. Let $\mu > \lambda$. Because of (A4), $G(x)^\mu$ has an ultimately increasing derivative. Therefore, if x is large and ε small enough

$$\begin{aligned} G(x)^\mu - G(x - \varepsilon g(x)G(x))^\mu &\leq \varepsilon g(x)G(x)\mu G(x)^{\mu-1}g(x)^{-1} \\ &\leq (1-2^{-\mu})G(x)^\mu, \end{aligned}$$

or

$$G(x - \varepsilon g(x)G(x)) \geq \tfrac{1}{2}G(x). \tag{3.1}$$

By further decreasing ε, if necessary, this holds for all $x \geq 2$. Now let $x \geq 2$ and $|a| \leq \varepsilon g(x)G(x)$. From Lemma 2 and (3.1), with a suitable b such that $|b| \leq |a|$,

$$\begin{aligned} |g(x+a) - g(x)| = |g'(x+b)| \cdot |a| &\leq C_1 \frac{|a|}{G(x+b)} \\ &\leq \frac{C_1|a|}{G(x - \varepsilon g(x)G(x))} \leq \frac{2C_1|a|}{G(x)}, \end{aligned}$$

thus

$$\left|\frac{g(x+a)}{g(x)} - 1\right| \leq \frac{2C_1|a|}{g(x)G(x)}. \tag{3.2}$$

This implies

$$|G(x+a) - G(x)| = \frac{|a|}{g(x+b)} \leq C_2\frac{|a|}{g(x)}$$

for a suitable $C_2>0$, thus

$$\left|\frac{G(x+a)}{G(x)}-1\right| \leqslant C_2 \frac{|a|}{g(x)G(x)}. \tag{3.3}$$

Combining (3.2) and (3.3) our claim follows. □

Next fix $\varepsilon>0$ and denote

$$Z_n = X_n + g(X_n),$$

$$\theta_n = \left[-\frac{\xi_{n+1}}{G(Z_n)^{1+\varepsilon}g(Z_n)} + \frac{\lambda+3\varepsilon}{2}\frac{\xi_{n+1}^2}{G(Z_n)^{2+\varepsilon}g(Z_n)^2}\right]$$
$$\times I(|\xi_{n+1}| \leqslant \varepsilon g(Z_n)G(Z_n),\, X_n>1)$$
$$+ I(|\xi_{n+1}| > \varepsilon g(Z_n)G(Z_n),\, X_n>1).$$

(I(A) denotes the indicator function of the set A.)

Lemma 4. *Under the assumptions of Theorem* 1, *if* $\varepsilon>0$ *is small and* $M>1$ *big enough, then for* X_n, $X_{n+1}\geqslant M$,

$$\varepsilon^{-1}Y_{n+1}^{-\varepsilon} \leqslant \varepsilon^{-1}Y_n^{-\varepsilon} + \theta_n - \mathbb{E}(\theta_n\,|\,F_n) - \varepsilon Y_n^{-1-\varepsilon}.$$

Proof. (i) Let U_n be any random variable such that

$$X_n \leqslant U_n \leqslant Z_n = X_n + g(X_n).$$

Choosing $\varepsilon = 1/G(X_n)$ in Lemma 3 we obtain for large X_n,

$$\left|\frac{g(U_n)}{g(X_n)}-1\right| + \left|\frac{G(U_n)}{Y_n}-1\right| \leqslant \frac{C}{Y_n}. \tag{3.4}$$

(ii) Next we estimate $\mathbb{E}(\theta_n\,|\,F_n)$. If $D>0$ is large enough

$$\theta_n \leqslant -\frac{\xi_{n+1}}{G(Z_n)^{1+\varepsilon}g(Z_n)} + \frac{\lambda+3\varepsilon}{2}\frac{\xi_{n+1}^2}{G(Z_n)^{2+\varepsilon}g(Z_n)^2}$$
$$+\frac{D\xi_{n+1}^2}{G(Z_n)^2 g(Z_n)^2} I(|\xi_{n+1}| > \varepsilon g(Z_n)G(Z_n)).$$

Using (A3) and (3.4) with $U_n = Z_n$, if X_n is large enough,

$$\mathbb{E}(\theta_n\,|\,F_n) \leqslant \frac{\lambda+3\varepsilon}{2}\frac{\sigma^2(X_n)}{G(Z_n)^{2+\varepsilon}g(Z_n)^2} + c\frac{\sigma(X_n)^\delta}{G(Z_n)^\delta g(Z_n)^\delta}$$
$$\leqslant \left(\frac{\lambda}{2}+2\varepsilon\right)\frac{\sigma^2(X_n)}{Y_n^{2+\varepsilon}g(X_n)^2} + c\frac{c(X_n)^\delta}{Y_n^\delta g(X_n)^\delta}.$$

By means of (1.1), if ε is small and X_n large enough,

$$\mathbb{E}(\theta_n \mid F_n) \leq \left(\frac{\lambda\beta}{2} + (2\beta+1)\varepsilon\right) Y_n^{-1-\varepsilon} + cY_n^{-\delta/2}$$
$$\leq \left(\frac{\lambda\beta}{2} + (2\beta+2)\varepsilon\right) Y_n^{-1-\varepsilon}.$$

Finally, since $\lambda\beta < 2$, if ε is sufficiently small,

$$\mathbb{E}(\theta_n \mid F_n) \leq (1-2\varepsilon) Y_n^{-1-\varepsilon}. \tag{3.5}$$

(iii) Next let $f(x) = \varepsilon^{-1} G(x)^{-\varepsilon}$ for $x > 1$. We show that for large X_n and X_{n+1},

$$f(X_{n+1}) \leq f(Z_n) + \theta_n. \tag{3.6}$$

If $|\xi_{n+1}| > \varepsilon g(Z_n) G(Z_n)$ and X_{n+1} is large enough,

$$f(X_{n+1}) \leq 1 = \theta_n \leq f(Z_n) + \theta_n.$$

Thus let $|\xi_{n+1}| \leq \varepsilon g(Z_n) G(Z_n)$. From a Taylor expansion

$$f(X_{n+1}) = f(Z_n) - \frac{\xi_{n+1}}{G(Z_n)^{\varepsilon+1} g(Z_n)} + \tfrac{1}{2} f''(V_n) \xi_{n+1}^2,$$

where, using (3.4) with $U_n = Z_n$,

$$|V_n - Z_n| \leq |\xi_{n+1}| \leq \varepsilon g(Z_n) G(Z_n) \leq 2\varepsilon g(X_n) Y_n,$$

if X_n is large enough. Now Lemma 3 entails

$$f''(V_n) = G(V_n)^{-2-\varepsilon} g(V_n)^{-2} (1 + \varepsilon + g'(V_n) G(V_n))$$
$$\leq G(V_n)^{-2-\varepsilon} g(V_n)^{-2} (\lambda + 2\varepsilon)$$
$$\leq G(Z_n)^{-2-\varepsilon} g(Z_n)^{-2} (\lambda + 3\varepsilon),$$

from which our claim follows. (Note that $\lambda \geq 0$ in view of Lemma 2.)

(iv) Finally, using (3.4), by means of a Taylor expansion with a suitable $X_n \leq U_n \leq Z_n$,

$$f(Z_n) = f(X_n) - G(U_n)^{-1-\varepsilon} \frac{g(X_n)}{g(U_n)}$$
$$\leq f(X_n) - (1-\varepsilon) Y_n^{-1-\varepsilon},$$

if X_n is large enough. This estimate together with (3.5) and (3.6) gives the desired result. □

Lemma 5. *Under the assumptions of Theorem* 1 *there is a* $\kappa < 1$ *such that* $\xi_{n+1} = o(g(X_n) Y_n^{\kappa})$ *a.s. on* $\mathscr{E}_\infty$.

Proof. Let

$$M_n = \sum_{k=0}^{n} (\theta_k - \mathbb{E}(\theta_k \mid F_k)).$$

M_n is a martingale, further by construction $|\theta_n| \leq C$ for a suitable $C > 0$. By means of a martingale theorem [3, Theorem 2.14] almost surely either $\liminf M_n = -\infty$ or M_n converges to a finite limit. Now there is a (random) N such that on the event $\mathscr{E}_\infty$ $X_n \geq M > 1$ for all $n \geq N$. From Lemma 4, if $n \geq N$,

$$\begin{aligned}0 &\leq \varepsilon^{-1} Y_{n+1}^{-\varepsilon} \leq \varepsilon^{-1} Y_N^{-\varepsilon} + M_n - M_{N-1} - \varepsilon \sum_{k=N}^{n} Y_k^{-1-\varepsilon} \\ &\leq \varepsilon^{-1} Y_N^{-\varepsilon} + M_n - M_{N-1}.\end{aligned}$$

Then $\liminf M_n = -\infty$ cannot occur, thus M_n has to converge a.s. on $\mathscr{E}_\infty$, and consequently

$$\sum_{k=N}^{\infty} Y_k^{-1-\varepsilon} < \infty \tag{3.7}$$

a.s. on $\mathscr{E}_\infty$. This holds for all $\varepsilon > 0$ sufficiently small, hence for all $\varepsilon > 0$. Now from (A3) and (1.1), with $0 < \kappa < 1$,

$$\Pr(|\xi_{n+1}| \geq g(X_n) Y_n^\kappa \mid F_n) \leq Cg(X_n)^{-\delta} Y_n^{-\kappa\delta} \sigma(X_n)^\delta \leq c Y_n^{\delta(1/2-\kappa)}.$$

If κ is close enough to 1, $\delta(\frac{1}{2} - \kappa) < -1$. Using (3.7),

$$\sum_{n=0}^{\infty} \Pr(|\xi_{n+1}| \geq g(X_n) Y_n^\kappa \mid F_n) < \infty$$

a.s. on $\mathscr{E}_\infty$. The martingale version of the Borel-Cantelli lemma yields that $|\xi_{n+1}| \geq g(X_n) Y_n^\kappa$ occurs only finitely often a.s. on $\mathscr{E}_\infty$, and the desired result follows. $\square$

4. Proof of Theorem 1

We apply the method of moments to a suitably truncated version of Y_n. Fix $B > 1$, $0 < \kappa < 1$ and $m \in \mathbb{N}$ and define

$$\begin{aligned}\tau &= \tau(m, \kappa, B) \\ &= \inf\{n \geq m: g(X_n) + |\xi_{n+1}| \geq g(X_n) Y_n^\kappa \text{ or } X_{n+1} \leq B\}.\end{aligned}$$

Note that τ is not a (F_n)-stopping time, however,

$$\{\tau \geq n\} \in F_n,$$

which is sufficient for our purposes.

Lemma 6. *If B is large enough and if κ is close enough to* 1, *there is a $\gamma > 0$ such that for every $0 \leq k \leq 2$ and $n \geq m$ a.s.*

$$\mathbb{E}\left(\frac{|\xi_{n+1}|^k}{g(X_n)^k} I(\tau(m, \kappa, B) = n) \,\middle|\, F_n \right) \leq I(\tau(m, \kappa, B) \geq n) Y_n^{k-1-\gamma}.$$

Note that $X_n > B$ on the event $\{\tau \geq n\}$.

Proof. First we analyse the event $\{\tau = n\}$. If $\tau = n$ and $g(X_n) + |\xi_{n+1}| < g(X_n) Y_n^\kappa$, then by definition of τ $X_{n+1} \leq B$, thus

$$X_n = X_{n+1} - g(X_n) - \xi_{n+1} \leq B + g(X_n) Y_n^\kappa.$$

In view of Lemma 2 $g(x)G(x)^\kappa = o(x)$, therefore

$$X_n \leq B + \tfrac{1}{2} X_n,$$

if B (and thus X_n) is large enough. It follows

$$\{\tau = n\} \subset \{\tau \geq n\} \cap \{g(X_n) + |\xi_{n+1}| \geq g(X_n) Y_n^\kappa \text{ or } B \leq X_n \leq 2B\}.$$

Now let $\delta > 2$ as in (A3). Then

$$\begin{aligned}
&\mathbb{E}(|\xi_{n+1}|^k \cdot I(\tau = n) \mid F_n) \\
&\quad\leq I(\tau \geq n) \mathbb{E}(|\xi_{n+1}|^k \cdot I(g(X_n) + |\xi_{n+1}| \geq g(X_n) Y_n^\kappa) \mid F_n) \\
&\qquad + I(\tau \geq n) \mathbb{E}(|\xi_{n+1}|^k \cdot I(|\xi_{n+1}| \leq g(X_n) Y_n^\kappa, B \leq X_n \leq 2B) \mid F_n) \\
&\quad\leq I(\tau \geq n)(g(X_n) Y_n^\kappa)^{k-\delta} \mathbb{E}((|\xi_{n+1}| + g(X_n))^\delta \mid F_n) \\
&\qquad + I(\tau \geq n) \max_{B \leq x \leq 2B} g(x) G(x)^\kappa \cdot I(B \leq X_n \leq 2B) \\
&\quad\leq I(\tau \geq n)(g(X_n) Y_n^\kappa)^{k-\delta} 2^\delta (C\sigma(X_n)^\delta + g(X_n)^\delta) \\
&\qquad + cI(\tau \geq n, B \leq X_n \leq 2B).
\end{aligned}$$

Using $\sigma^2(x) = O(g(x)^2 G(x))$, if κ is close enough to 1 and B is large enough, for a suitable $\gamma > 0$,

$$\begin{aligned}
&\mathbb{E}(|\xi_{n+1}|^k \cdot I(\tau = n) \mid F_n) \\
&\quad\leq cI(\tau \geq n)\{g(X_n)^k Y_n^{\kappa(k-\delta)+\delta/2} + I(B \leq X_n \leq 2B)\} \\
&\quad\leq cI(\tau \geq n) g(X_n)^k Y_n^{k-1-\gamma}. \qquad \square
\end{aligned}$$

Lemma 7. *If B is large enough and κ close enough to 1, then for any $m \geq 1$, $\eta > 0$ and $\alpha = 0, 1, \ldots,$*

$$\begin{aligned}
&\lim_{n \to \infty} n^{-\alpha} \mathbb{E}(Y_n^\alpha \cdot I(\tau(m, \kappa, B) \geq n, X_m \leq \eta)) \\
&\quad= c_\alpha \Pr(\tau(m, \kappa, B) = \infty, X_m \leq \eta),
\end{aligned}$$

where $c_\alpha = (1 + \tfrac{1}{2}\beta(\alpha - \lambda)) c_{\alpha-1}$, $c_0 = 1$.

Proof. The proof proceeds by induction with respect to α, the case $\alpha = 0$ being obvious. Consider the Taylor expansion

$$\begin{aligned}
G(X_{n+1})^\alpha &= G(X_n)^\alpha + \alpha g(X_n)^{-1} G(X_n)^{\alpha-1}(g(X_n) + \xi_{n+1}) \\
&\quad + \tfrac{1}{2}\alpha G(V_n)^{\alpha-2} g(V_n)^{-2}(\alpha - 1 - g'(V_n)G(V_n))(g(X_n) + \xi_{n+1})^2
\end{aligned}$$

with $|V_n - X_n| \leq |X_{n+1} - X_n| \leq g(X_n) + |\xi_{n+1}|$. Now $\tau = \tau(m, \kappa, B) \geq n+1$ implies $X_n > B$ and $|V_n - X_n| \leq g(X_n) Y_n^\kappa$. In view of Lemma 3, if B is large enough,

$$\left|\frac{G(V_n)}{G(X_n)} - 1\right| + \left|\frac{g(V_n)}{g(X_n)} - 1\right| \leq C Y_n^{\kappa-1},$$

such that we may rewrite our Taylor expansion as

$$\begin{aligned} Y_{n+1}^\alpha \cdot I(\tau \geq n+1) = {} & Y_n^\alpha \cdot I(\tau \geq n+1) \\ & + \alpha Y_n^{\alpha-1}\left(1 + \frac{\xi_{n+1}}{g(X_n)}\right) \cdot I(\tau \geq n+1) \\ & + \tfrac{1}{2}\alpha(\alpha-\lambda) Y_n^{\alpha-2}\left(1 + \frac{\xi_{n+1}}{g(X_n)}\right)^2 (1+U_n) \cdot I(\tau \geq n+1) \end{aligned}$$

with

$$|U_n| \leq f_1(Y_n)$$

and $f_1(x) = \mathrm{o}(1)$, as $x \to \infty$. Since $I(\tau \geq n+1) = I(\tau \geq n) - I(\tau = n)$,

$$\begin{aligned} & \mathbb{E}(Y_{n+1}^\alpha \cdot I(\tau \geq n+1) \mid F_n) \\ & \quad = Y_n^\alpha \cdot I(\tau \geq n) + \alpha Y_n^{\alpha-1} \cdot I(\tau \geq n) \\ & \qquad + \tfrac{1}{2}\alpha(\alpha-\lambda) Y_n^{\alpha-2} \cdot I(\tau \geq n) \mathbb{E}\left(\left(1 + \frac{\xi_{n+1}}{g(X_n)}\right)^2 (1+U_n) \,\middle|\, F_n\right) + W_n, \end{aligned}$$

where

$$|W_n| \leq c Y_n^{\alpha-1-\gamma} \cdot I(\tau \geq n)$$

in view of Lemma 6. Since

$$\mathbb{E}\left(\left(1 + \frac{\xi_{n+1}}{g(X_n)}\right)^2 \,\middle|\, F_n\right) = 1 + \sigma^2(X_n) g(X_n)^{-2} \sim \beta Y_n,$$

this simplifies to

$$\begin{aligned} & \mathbb{E}(Y_{n+1}^\alpha \cdot I(\tau \geq n+1) \mid F_n) \\ & \quad = Y_n^\alpha \cdot I(\tau \geq n) + \alpha(1 + \tfrac{1}{2}\beta(\alpha-\lambda)) Y_n^{\alpha-1} \cdot I(\tau \geq n)(1+R_n) \end{aligned}$$

with

$$|R_n| \leq f_2(Y_n)$$

and $f_2(x) = \mathrm{o}(1)$. Taking expectations on the event $\{X_m \leq \eta\}$, for $n \geq m$,

$$\begin{aligned} & \mathbb{E}(Y_{n+1}^\alpha \cdot I(\tau \geq n+1, X_m \leq \eta)) \\ & \quad = \mathbb{E}(Y_n^\alpha \cdot I(\tau \geq n, X_m \leq \eta)) \\ & \qquad + \alpha(1 + \tfrac{1}{2}\beta(\alpha-\lambda)) \mathbb{E}(Y_n^{\alpha-1} \cdot I(\tau \geq n, X_m \leq \eta))(1+\mathrm{o}(1)). \end{aligned}$$

Now assume that

$$\mathbb{E}(Y_n^{\alpha-1}\cdot I(\tau\geq n, X_m\leq\eta))\sim c_{\alpha-1}n^{\alpha-1}\Pr(\tau=\infty, X_m\leq\eta).$$

Then

$$\begin{aligned}
&\mathbb{E}(Y_{n+1}^{\alpha}\cdot I(\tau\geq n+1, X_m\leq\eta))\\
&\quad=\mathbb{E}(Y_n^{\alpha}\cdot I(\tau\geq n, X_m\leq\eta))\\
&\qquad+\alpha(1+\tfrac{1}{2}\beta(\alpha-\lambda))c_{\alpha-1}n^{\alpha-1}\Pr(\tau=\infty, X_m\leq\eta)(1+\mathrm{o}(1))\\
&\quad=\mathbb{E}(Y_m^{\alpha}\cdot I(\tau\geq m, X_m\leq\eta))\\
&\qquad+\alpha(1+\tfrac{1}{2}\beta(\alpha-\lambda))c_{\alpha-1}\Pr(\tau=\infty, X_m\leq\eta)\sum_{k=m}^{n}k^{\alpha-1}(1+\mathrm{o}(1))\\
&\quad=c_\alpha\Pr(\tau=\infty, X_m\leq\eta)n^{\alpha}(1+\mathrm{o}(1)),
\end{aligned}$$

and the induction is finished. □

Proof of Theorem 1. As we noted in the second comment in section 1, we may apply Theorem 2(ii) of [7], therefore

$$\Pr\Big(X_n\to\infty \text{ or } \limsup_{n\to\infty} X_n<B\Big)=1,$$

if $B>1$ is chosen large enough. In view of Lemma 5, if κ is close enough to 1,

$$\lim_{m\to\infty}\Pr(\{\tau(m,\kappa,B)=\infty\}\triangle\mathscr{E}_\infty)=0, \tag{4.1}$$

where $\triangle$ denotes the symmetric difference of events. Now, since $\{\tau\geq n\}\uparrow\{\tau=\infty\}$, it follows from Lemma 7,

$$n^{-\alpha}\mathbb{E}(Y_n^{\alpha}\,|\,\tau\geq n, X_m\leq\eta)=c_\alpha+\mathrm{o}(1)$$

for $\alpha=0,1,\ldots$. Since the c_α are the moments of the density

$$f(x)=(\tfrac{1}{2}\beta)^{\lambda-2\beta-1}\Gamma(2/\beta-\lambda+1)^{-1}x^{2/\beta-\lambda}\exp\left(-\frac{2x}{\beta}\right),\quad x\geq 0,$$

an application of the method of moments yields

$$\Pr(Y_n\leq tn\,|\,\tau\geq n, X_m\leq\eta)\to\int_0^t f(x)\,\mathrm{d}x.$$

Letting $\eta\to\infty$ it is not difficult to conclude that

$$\Pr(Y_n\leq tn\,|\,\tau(m,\kappa,B)=\infty)\to\int_0^t f(x)\,\mathrm{d}x,$$

and letting $m\to\infty$, our claim follows by means of (4.1). □

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